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Central Subspaces of Banach Spaces

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In this paper we study Banach spaces that admit weighted Chebyshev centres for finite sets. Such spaces have been extensively studied recently by Veselý using the approach of finitely intersecting balls. Following his approach we exhibit large classes of Banach spaces that have this property. Certain stability results for spaces of vector valued continuous and Bochner integrable functions are also obtained.

Key Words: central subspace; Chebyshev centre; generalized centre (GC); L^1-predual, proximinal subspace; Bochner L^p space.

1. INTRODUCTION

Let X be a Banach space. We will denote by $B_X[x, r]$ (or $B[x, r]$, if there is no scope for confusion) the closed ball of radius $r > 0$ around $x \in X$. We will identify an element $x \in X$ with its canonical image in X^{**}. $B_{X^{**}}[x, r]$ will be denoted by $B^{**}[x, r]$. All subspaces we consider are norm closed. Our notations are otherwise standard. Any unexplained terminology can be found in either [3] or [5].

In this paper, we study Banach spaces that admit weighted Chebyshev centres for finite sets. Our motivation comes from the recent work of [20] (We take this opportunity to thank Professor Veselý for sending us the preprint in April, 1997). We state a key result from [20].

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Let X be a Banach space. Let $\{a_1, a_2, ..., a_n\} \subseteq X$. Let $f\colon \mathbb{R}^n_+ \to \mathbb{R}$. Minimizers of the function $\phi\colon X \to \mathbb{R}$ defined by

$$\phi(x) = f(\|x - a_1\|, \|x - a_2\|, ..., \|x - a_n\|), \tag{1}$$

are called f-centres of $\{a_1, a_2, ..., a_n\}$. If f is of the form

$$f(t_1, t_2, ..., t_n) = \max_{1 \leqslant i \leqslant n} r_i t_i,$$

with $r_1, r_2, ..., r_n > 0$, the f-centres are called weighted Chebyshev centres.

THEOREM 1.1 [20, Theorem 2.7]. *For a Banach space X and $a_1, a_2, ..., a_n \in X$, the following are equivalent*:

(a) *If $r_1, r_2, ..., r_n > 0$ and $\bigcap_{i=1}^n B^{**}[a_i, r_i] \neq \emptyset$, then $\bigcap_{i=1}^n B[a_i, r_i] \neq \emptyset$.*

(b) *$\{a_1, a_2, ..., a_n\}$ admits weighted Chebyshev centres for all weights $r_1, r_2, ..., r_n > 0$.*

(c) *$\{a_1, a_2, ..., a_n\}$ admits f-centres for every continuous monotone coercive $f\colon \mathbb{R}^n_+ \to \mathbb{R}$ (See [20] for the definitions).*

DEFINITION 1.2 [20, Definition 2.8]. A Banach space X is said to belong to the class (GC), denoted $X \in (GC)$, if for every $n \in \mathbb{N}$ and $a_1, a_2, ..., a_n \in X$, the three equivalent conditions of Theorem 1.1 are satisfied.

The main aim of this paper is to exhibit several classes of Banach spaces that belong to the class (GC) and to explore its connections with other intersection properties of balls studied in the literature. For this reason most often we work with condition (a) of Theorem 1.1. We note that membership of the class (GC) is a finite version of the finite intersection property (FIP) studied by Godefroy in [4].

It is well known that several aspects of optimal estimation are connected to the ball related geometry of the underlying space (see [6, Section 33]). This connection is explored in Section 3 where, using an intersection of balls argument, we show that any compact subset of a Banach space whose dual is isometric to a $L^1(\mu)$ space (μ is a positive measure), is centrable (see Definition 3.8).

In Section 4, we study the stability of the class (GC), which in turn gives more examples of spaces that belong to this class. We also consider in this section, quotient spaces and spaces of vector valued continuous and Bochner integrable functions and study the stability aspect. This highlights the strong interrelation between the approximation theoretic concepts and purely functional analytic and measure theoretic concepts such as Radon–Nikodým Property (RNP), universal measurability (see [2, Chapter 7, Section 8]) etc.

We now give a more detailed account of our results.

In Section 2, we consider some general result about the class (GC). To facilitate this, we define a central subspace (C-subspace) of a Banach space by a relative intersection property of balls (see Definition 2.1) and observe that a Banach space X belongs to the class (GC) if and only if it is a C-subspace of some dual space. We show that 1-complemented subspaces and semi-L-summands are C-subspaces. We also obtain an intrinsic characterization of the class (GC).

In Section 3, apart from the results mentioned above, we also show that a Banach space whose dual is isometric to a $L^1(\mu)$ space belongs to the class (GC). This follows as a consequence of our characterization of (GC) spaces as those Banach spaces which are C-subspaces in some superspace that is a dual.

In Section 4, we only consider Banach spaces over real scalars. Using the equivalent condition (b) of Theorem 1.1, we show that membership of the class (GC) is a separably determined property. An easy consequence of our result on transitivity of C-subspaces is that if $X \in (GC)$ and $Y \subseteq X$ is a reflexive subspace then $X/Y \in (GC)$. For a compact Hausdorff space K and a finite dimensional space X, we show that if the space $C(K, X)$ of X-valued continuous functions on K belong to the class (GC), it has the intersection property $n.k.$IP (see Definition 2.10), for $k = \dim X + 1$ and any $n > k$. As a consequence of some recent work of Ehrhard Behrends, we observe that membership of the class (GC) is not a 3-space property, i.e., for a subspace $Y \subseteq X$, $Y \in (GC)$ and $X/Y \in (GC)$ need not imply that $X \in (GC)$. For the class of Bochner integrable functions, we show that if Y is a separable C-subspace of X, then for each $1 \leqslant p < \infty$, $L^p(\mu, Y)$ is a C-subspace of $L^p(\mu, X)$, and use this to show that if Y is a separable C-subspace of a dual space with the RNP, then $L^p(\mu, Y) \in (GC)$. We also note that for a Banach space X having the 3.2.IP, $L^1(\mu, X)$ has Chebyshev centres for sets of 3 elements.

2. CENTRAL SUBSPACES AND THE CLASS (GC)

In this section and the next, we consider spaces over real or complex scalars. We now base a definition on condition (a) of Theorem 1.1.

Definition 2.1. Let X be a Banach space. We say that a subspace $Y \subseteq X$ is a central subspace (C-subspace) of X if every finite family of closed balls with centres in Y that intersects in X also intersects in Y.

In particular, $X \in (GC)$ if and only if X is a C-subspace of X^{**}.

We summarize in Proposition 2.2, certain observations regarding C-subspaces.

PROPOSITION 2.2. (a) *Y is a C-subspace of X if and only if for any a_1, $a_2, ..., a_n \in Y$ and $x \in X$, there exists $y \in Y$ such that $\|y - a_i\| \leqslant \|x - a_i\|$ for all $i = 1, 2, ..., n$.*

(b) *1-complemented subspaces are C-subspaces.*

(c) *Z is a C-subspace of Y and Y is a C-subspace of X implies Z is a C-subspace of X.*

(d) *$X \in (GC)$ if and only if X is a C-subspace of some dual space.*

(e) *the membership of the class (GC) is inherited by C-subspaces, in particular, by 1-complemented subspaces.*

Proof. The first result that needs a proof is (d).

Let X be a C-subspace of some Z^*. Then $X \subseteq Z^*$ implies $X^{**} \subseteq Z^{***}$. If a finite family of balls centred at points of X intersect in X^{**}, they intersect in Z^{***}. Since Z^* is 1-complemented in Z^{***}, by (b), these balls intersect in Z^*. Since X is a C-subspace of Z^*, they intersect in X too.

The proof of (e) follows from (c) and (d). ∎

Remark 2.3. The observation that membership of the class (GC) is inherited by 1-complemented subspaces allows one to simplify many of the arguments in [20].

Since the membership of the class (GC) is inherited by C-subspaces, we would like to identify C-subspaces beyond 1-complemented subspaces. In this context, we recall the following definition due to Lima.

DEFINITION 2.4 [5, page 43]. A subspace Y of a Banach space X is called a semi-L-summand if there exists a (not necessarily linear) projection P: $X \to Y$ such that

$$P(\lambda x + Py) = \lambda Px + Py, \qquad \text{and}$$

$$\|x\| = \|Px\| + \|x - Px\|$$

for all $x, y \in X$, $\lambda \in \mathbb{K}$.

PROPOSITION 2.5. *A semi-L-summand is a C-subspace.*

Proof. Let Y be a semi-L-summand in X. Let $y_1, y_2, ..., y_n \in Y$ and $x \in X$. Let P be as in the definition. Then

$$\|y_i - Px\| \leqslant \|y_i - Px\| + \|x - Px\| = \|y_i - x\|$$

for all $i = 1, 2, ..., n$. Hence by Proposition 2.2(a), we get that Y is a C-subspace of X. ∎

Another class of subspaces that it is natural to consider in this context are M-ideals.

DEFINITION 2.6 [5]. A subspace $M \subseteq X$ is said to be a M-ideal if there exists a subspace $N \subseteq X^*$ such that $X^* = M^\perp \oplus_1 N$.

QUESTIONS 2.7. *Is an M-ideal a C-subspace? Is the membership of the class (GC) inherited by M-ideals?*

PROPOSITION 2.8. *If $M \subseteq X$ is a M-ideal and $M \in (GC)$, then M is a C-subspace of X.*

Proof. Let $\{m_1, m_2, ..., m_k\} \subseteq M$ and $x \in X$. Since $X^{**} = M^{\perp\perp} \oplus_\infty N^\perp$, we can write $x = y + z$, where $y \in M^{\perp\perp}$ and $z \in N^\perp$. Since $M \in (GC)$, identifying M^{**} with $M^{\perp\perp}$, we get an $m \in M$ such that $\|m - m_i\| \leqslant \|y - m_i\|$ for all i. Note that $\|x - m_i\| = \max\{\|y - m_i\|, \|z\|\}$. Hence M is a C-subspace of X. ∎

In the study of intersection properties of balls, sometimes one can only prove that balls with slightly larger radii intersect. One standard trick in such a case is to work in a suitable dual space, and use the weak* compactness of closed balls to show that the original set of balls intersect. We now obtain an intrinsic characterisation of the class (GC) that shows that this class is a natural setting for similar "compactness arguments".

PROPOSITION 2.9. *A Banach space $X \in (GC)$ if and only if for all $n \in \mathbb{N}$, $a_1, a_2, ..., a_n \in X$ and $r_1, r_2, ..., r_n > 0$, $\bigcap_{i=1}^n B[a_i, r_i + \varepsilon] \neq \emptyset$ for all $\varepsilon > 0$ implies $\bigcap_{i=1}^n B[a_i, r_i] \neq \emptyset$.*

Proof. The "if" part follows from a simple form of the Principle of Local Reflexivity as in [11, Lemma 5.8].

"Only if" part: Suppose $X \in (GC)$ and let $\{B[x_i, r_i]: i = 1, ..., n\}$ be a family of closed balls in X such that for every $\varepsilon > 0$,

$$\bigcap_{i=1}^{n} B[x_i, r_i + \varepsilon] \neq \emptyset.$$

Consider the family $\{B^{**}[x_i, r_i + \varepsilon]: i = 1, ..., n; \varepsilon > 0\}$ in X^{**}. Then any finite subfamily intersects. Hence, by w*-compactness,

$$\bigcap_{i=1}^{n} B^{**}[x_i, r_i] \neq \emptyset$$

and so

$$\bigcap_{i=1}^{n} B[x_i, r_i] \neq \varnothing. \quad \blacksquare$$

And here is an application of the idea. We need the following definition.

DEFINITION 2.10 [10]. A Banach space X has the *a.n.k.*IP if, for each $\varepsilon > 0$, and each family $\{B[x_i, r_i]: i = 1, ..., n\}$ of n closed balls such that any k of them intersect, we have

$$\bigcap_{i=1}^{n} B[x_i, r_i + \varepsilon] \neq \varnothing.$$

If we can take $\varepsilon = 0$, we say that X has the *n.k.*IP.

COROLLARY 2.11. *A Banach space X belonging to the class (GC) has the a.n.k.IP for some n, k if and only if it has the n.k.IP.*

DEFINITION 2.12. A Banach space X has the finite intersection property (FIP) if every family of closed balls with centres in X that intersects in X^{**} also intersects in X.

Thus the FIP is the infinite analog of the membership of the class (GC) (see [4, 15]). It is clear that a space with the FIP belongs to the class (GC). It is easy to see that a Banach space X has the FIP if and only if every finitely intersection family of closed balls in X has non-empty intersection. It follows from the results of [20] that c_0, the space of sequences converging to zero, is in the class (GC). But it is well known that c_0 lacks the FIP.

3. L^1-PREDUALS

We now exhibit a large class of Banach spaces that belong to the class (GC). Since $c_0 \in (GC)$ and $c_0^* = \ell^1$, it is natural to ask if every Banach space whose dual is isometric to a $L^1(\mu)$ space is in this class.

DEFINITION 3.1 [9]. A Banach space X whose dual X^* is isometrically isomorphic to $L^1(\mu)$ for some positive measure μ is called an L^1-predual.

We show that an L^1-predual indeed belongs to the class (GC) by using the characterizations of L^1-preduals in terms of intersection properties of balls as obtained in [7] and [11]. We need a definition from [7].

DEFINITION 3.2. A family $\{B_X[x_i, r_i]\}$ of closed balls is said to have the weak intersection property if for all $x^* \in B_{X^*}[0, 1]$, the family $\{B[x^*(x_i), r_i]\}$ intersects in $\mathbb{K}$.

THEOREM 3.3. *A Banach space X is an L^1-predual if and only if whenever X is a subspace of a dual space, it is a C-subspace there.*

Proof. Suppose whenever $X \subseteq Y^*$, X is a C-subspace of Y^*.

It is well known that $X \subseteq \ell^\infty(\Gamma)$ for some discrete set Γ. By hypothesis, X is a C-subspace of $\ell^\infty(\Gamma)$. By [7, Theorem 4.9], X is an L^1-predual if and only if any finite family $\{B_X[x_i, r_i]\}$ of closed balls with the weak intersection property intersects in X.

If $\{B_X[x_i, r_i]\}$ is such a family, then considering the evaluation functional at $\gamma \in \Gamma$,

$$\bigcap_{i=1}^{n} B_{\mathbb{K}}[x_i(\gamma), r_i] \neq \emptyset.$$

And hence

$$\bigcap_{i=1}^{n} B_{\ell^\infty(\Gamma)}[x_i, r_i] \neq \emptyset.$$

Since X is a C-subspace of $\ell^\infty(\Gamma)$,

$$\bigcap_{i=1}^{n} B_X[x_i, r_i] \neq \emptyset.$$

Conversely, suppose X is an L^1-predual, and let $X \subseteq Y^*$. Let $\{x_i \in X, r_i \geqslant 0 \text{ for } i = 1, ..., n\}$, $n \geqslant 3$ be such that

$$\bigcap_{i=1}^{n} B_{Y^*}[x_i, r_i] \neq \emptyset$$

Hence this family of balls weakly intersects in Y^*. Since the centres of the balls are in X, we conclude that $\{B_X[x_i, r_i]: i = 1, ..., n\}$ has the weak intersection property. Therefore by [7] again,

$$\bigcap_{i=1}^{n} B_X[x_i, r_i] \neq \emptyset.$$

For $n = 2$, observe that two balls intersect if and only if the distance between the centres is less than or equal to the sum of the radii. Therefore, it is independent of the ambient space. ∎

Corollary 3.4. *Every L^1-predual belongs to the class (GC).*

Remark 3.5. It is well known that an L^1-predual has the FIP if and only if it is isometric to $C(K)$ for some extremally disconnected compact Hausdorff space K (see [9, Chapter 3]).

A proper M-ideal cannot have the FIP [15], but may belong to the class (GC), as the following example shows.

Example 3.6. Let $I = \{f \in C[0,1]: f([0,1/2]) = 0\}$. Then I is a proper M-ideal [1, p. 13] and an L^1-predual [9, p. 218, Exercise 7], and hence it belongs to the class (GC).

Remark 3.7. In fact, L^1-preduals are the proper domains to consider even stronger forms of optimal estimations. Our next theorem illustrates this and extends the corollary on p. 194 of [6], since a $\mathscr{P}_1$-space is an L^1-predual. We also note that a Banach space X is an L^1-predual if and only if X^{**} is isometric to a $\mathscr{P}_1$-space (see [10, Theorem 4.1] for the complex case. The real case follows from [9] or [10]). In order to emphasize the clear connection with the class (GC), we further note that a Banach space is a $\mathscr{P}_1$-space if and only if it is 1-complemented in every superspace [6, Section 33 (g)]. We also note the results that we will use from [6] are also valid for spaces over $\mathbb{K}$.

Let X be a Banach space. For a bounded subset $A \subseteq X$, observe that

$$r(A) \stackrel{\text{defn}}{=} \inf_{x \in X} \sup_{a \in A} \|x - a\| \geqslant \tfrac{1}{2} \operatorname{diam} A. \tag{2}$$

Definition 3.8 [6, Section 33 (g)]. A bounded subset $A \subseteq X$ is said to be centrable if equality holds in (2).

Theorem 3.9. *Let X be an L^1-predual. Then any compact set $A \subseteq X$ is centrable.*

Proof. Let $A \subseteq X$ be a compact set. Since X is an L^1-predual, X^{**} is a $\mathscr{P}_1$-space and hence, by [6, p. 193], A is centrable in X^{**}. Observe that the diameter of A in the two spaces X and X^{**} is the same. Since X^{**} is a dual space, arguing as in [20], there exists $x_o^{**} \in X^{**}$ such that

$$r^{**}(A) = \inf_{x^{**} \in X^{**}} \sup_{a \in A} \|x^{**} - a\| = \sup_{a \in A} \|x_o^{**} - a\| = \tfrac{1}{2} \operatorname{diam} A.$$

Then the family of balls $\{B_X[a, \|x^{**} - a\|]: a \in A\}$ have the weak intersection property. Since X is an L^1-predual, since the centres of the balls are in a compact set A, by [10, Proposition 4.4],

$$\bigcap_{a \in A} B_X[a, \|x_o^{**} - a\|] \neq \emptyset$$

If $x_o \in \bigcap_{a \in A} B_X[a, \|x_o^{**} - a\|]$,

$$r(A) \leqslant \sup_{a \in A} \|x_o - a\| \leqslant \sup_{a \in A} \|x_o^{**} - a\| = \tfrac{1}{2} \operatorname{diam} A.$$

Thus A is centrable. ▮

4. STABILITY RESULTS

In this section, we work only with real scalars. We begin our results on stability properties by proving that the class (GC) is separably determined.

PROPOSITION 4.1. *Membership of the class (GC) is a separably determined property, i.e., if for every separable subspace $Y \subseteq X$, $Y \in (GC)$, then $X \in (GC)$.*

Proof. Let every separable subspace of X belong to the class (GC). Let $\{a_1, a_2, ..., a_n\} \subseteq X$. Let $r_1, r_2, ..., r_n > 0$. Let $\phi: X \to \mathbb{R}$ be defined by

$$\phi(x) = \max_{1 \leqslant i \leqslant n} r_i \|x - a_i\|$$

By Theorem 1.1(b), it suffices to show that ϕ attains its minimum in X.

Let $\{x_m\}$ be a minimizing sequence for ϕ, i.e., $\inf \phi(X) = \lim_m \phi(x_m)$.

Let $Y = \overline{\operatorname{span}}[\{x_m\} \cup \{a_1, a_2, ..., a_n\}]$. Then Y is separable, $\{a_1, a_2, ..., a_n\} \subseteq Y$ and $\inf \phi(Y) = \inf \phi(X)$ (since $\{x_m\} \subseteq Y$). Since $Y \in (GC)$, by Theorem 1.1(b), ϕ attains its minimum in Y. Thus ϕ attains its minimum in X as well. ▮

If X is an L^1-predual and Z is a separable subspace of X, it is known [9, p. 227, Lemma 6] that then there exist a separable Y such that $Z \subseteq Y \subseteq X$ and Y is an L^1-predual. Since L^1-preduals belong to the class (GC), it is natural to ask:

QUESTION 4.2. *Suppose $X \in (GC)$ and Z is a separable subspace of X. Does there exist a separable Y such that $Z \subseteq Y \subseteq X$ and $Y \in (GC)$?*

Coming to quotient spaces, we have the following result.

DEFINITION 4.3. Let X be a Banach space. A subspace $Y \subseteq X$ is called proximinal if every $x \in X$ has a best approximant in Y, i.e., there exists $y_o \in Y$ such that $\|x - y_o\| = d(x, Y) = \inf_{y \in Y} \|x - y\|$.

PROPOSITION 4.4. *Let $Z \subseteq Y \subseteq X$, Z proximinal in X and Y is a C-subspace of X. Then Y/Z is a C-subspace of X/Z.*

Proof. Let $[y_1], [y_2], ..., [y_n] \in Y/Z$ and $[x] \in X/Z$. By proximinality of Z in X, for every $i = 1, 2, ..., n$, $\|[y_i] - [x]\| = d(y_i - x, Z) = \|y_i - x - z_i\|$ for some $z_i \in Z$. Since Y is a C-subspace of X, there exists $y_o \in Y$ such that $\|y_i - z_i - y_o\| \leqslant \|y_i - x - z_i\|$ for all $i = 1, 2, ..., n$. Clearly then $\|[y_i] - [y_o]\| \leqslant \|[y_i] - [x]\|$ for all $i = 1, 2, ..., n$. ∎

Since a reflexive subspace is proximinal, the following corollary is immediate from Proposition 2.2.

COROLLARY 4.5. *Let $X \in (GC)$ and let $M \subseteq X$ be a reflexive subspace. Then $X/M \in (GC)$.*

COROLLARY 4.6. *Let $Z \subseteq Y \subseteq X$, Z proximinal in Y and Y is a semi-L-summand in X. Then, Y/Z is a C-subspace of X/Z.*

Proof. From [16, Proposition 2], it follows that Z is proximinal in X. The result now follows from Propositions 2.5 and 4.4. ∎

Let us now consider the c_0 and ℓ_p sums.

THEOREM 4.7. *Let Γ be an index set. The c_0 or ℓ_p $(1 \leqslant p \leqslant \infty)$ sum of Y_α's is a C-subspace of the (resp.) c_0 or ℓ_p sum of X_α's if and only if Y_α is a C-subspace of X_α for all $\alpha \in \Gamma$.*

Proof. Let us denote by X and Y resp. the c_0 or ℓ_p $(1 \leqslant p \leqslant \infty)$ sum of X_α's and Y_α's.

Suppose Y is a C-subspace of X and let $\alpha_0 \in \Gamma$. Let $x_{\alpha_0} \in X_{\alpha_0}$, $y_{\alpha_0 1}$, $y_{\alpha_0 2}$, ..., $y_{\alpha_0 n} \in Y_{\alpha_0}$. Define $x \in X$ and $y_1, y_2, ..., y_n \in Y$, by putting 0 at every other coordinate. Then there exists $y \in Y$ such that $\|y - y_k\| \leqslant \|x - y_k\|$ for all $k = 1, 2, ..., n$. Let y_{α_0} be the α_0th coordinate of y. Clearly, $\|y_{\alpha_0} - y_{\alpha_0 k}\| \leqslant \|y - y_k\| \leqslant \|x - y_k\| = \|x_{\alpha_0} - y_{\alpha_0 k}\|$ for all $k = 1, 2, ..., n$.

Let Γ be an index set and for each $\alpha \in \Gamma$, let Y_α be a C-subspace of X_α. Let $x \in X$ and $y_1, y_2, ..., y_n \in Y$. For any $\alpha \in \Gamma$, there exists $y_\alpha \in Y_\alpha$ such that $\|y_\alpha - y_{\alpha k}\| \leqslant \|x_\alpha - y_{\alpha k}\|$ for all $k = 1, 2, ..., n$. By taking 0 as an additional centre we can also have $\|y_\alpha\| \leqslant \|x_\alpha\|$. Clearly, y defined with these coordinates belongs to Y and satisfies $\|y - y_k\| \leqslant \|x - y_k\|$ for all $k = 1, 2, ..., n$. ∎

COROLLARY 4.8. *The class* (GC) *is stable under* ℓ_p *sums* $(1 \leqslant p \leqslant \infty)$.

Proof. By Proposition 2.2(d), $X_\alpha \in (GC)$ if and only if X_α is a C-subspace of some Y_α^*. Now the ℓ_p sum of Y_α^*'s is a dual space, and hence, by Proposition 2.2(d) again, $X \in (GC)$. ▮

Remark 4.9. This has already been noted in [20] with a quite different proof.

It is easy to see that the above proof also works in the setting of Section 4 of [20] and shows that: $(\bigoplus Y_\alpha)_V$ is a C-subspace of $(\bigoplus X_\alpha)_V$ (see [20] for the notation) if and only if Y_α is a C-subspace of X_α for all $\alpha \in \Gamma$.

In [20], the author uses Theorem 1.1(b) to show that the class (GC) is also stable under c_0-sums. Here, in order to use the C-subspace argument, one needs to show that the c_0-sum of X_α's is a C-subspace of the ℓ_∞-sum of X_α's. We do not have a proof of this yet.

In passing, we note

PROPOSITION 4.10. *Let* $M, N \subseteq X$ *be two* M*-ideals in the class* (GC) *such that* $M \cap N$ *is reflexive. Then* $M + N/M \cap N$ *is in the class* (GC).

Proof. It follows from [1, Proposition 2.7] that $M+N$ is a closed subspace (in fact a M-ideal). From [1, Proposition 2.8], we get that $M+N/M \cap N$ is a ℓ^∞ direct sum of $M/M \cap N$ and $N/M \cap N$. Since $M \cap N$ is reflexive, both these component spaces are in the class (GC) by Corollary 4.5. Hence $M+N/M \cap N \in (GC)$, by Corollary 4.8. ▮

In [20], the author has analysed in detail the spaces X for which the space $C_b(T, X)$ of bounded X-valued continuous functions belongs to the class (GC) for every topological space T. Here we concentrate on $C(K, X)$, where K is a compact Hausdorff space. It is easy to see that for a topological space T and $Y \subseteq X$, if $C_b(T, Y)$ is a C-subspace of $C_b(T, X)$, then Y is a C-subspace of X. In the next Proposition, we prove a partial converse when Y is finite dimensional and K is extremally disconnected.

THEOREM 4.11. *Let* Y *be a finite dimensional* C*-subspace of a Banach space* X. *Then for any extremally disconnected compact Hausdorff space* K, $C(K, Y)$ *is a* C*-subspace of* $C(K, X)$.

Proof. Let K be homeomorphically embedded in the Stone-Čech compactification $\beta(S)$ of a discrete set S and let $\phi: \beta(S) \to K$ be a continuous retract (see [9]). Fix $f_1, f_2, ..., f_n \in C(K, Y)$ and $g \in C(K, X)$. Note that since Y is finite dimensional, any Y-valued bounded function on S has a norm preserving extension in $C(\beta(S), Y)$, by the defining property of $\beta(S)$. Thus $C(\beta(S), Y)$ is onto isometric to $\bigoplus_\infty Y$ (direct sum taken over

$|S|$-many copies of Y). In view of Theorem 4.7, this space is a C-subspace of $\bigoplus_\infty X$. This latter space contains $C(\beta(S), X)$. Hence $C(\beta(S), Y)$ is a C-subspace of $C(\beta(S), X)$. Now $g \circ \phi \in C(\beta(S), X)$ and $f_i \circ \phi \in C(\beta(S), Y)$ for all i. Thus there is a $h \in C(\beta(S), Y)$ such that $\|h - f_i \circ \phi\| \leqslant \|g \circ \phi - f_i \circ \phi\|$ for all i. Let $f = h|_K \in C(K, Y)$. Since ϕ is a retract, we conclude that $\|f_i - f\| \leqslant \|f_i - g\|$. Thus, $C(K, Y)$ is a C-subspace of $C(K, X)$. ∎

Remark 4.12. Rao [15] proved that for a finite dimensional X and an extremally disconnected compact Hausdorff space K, $C(K, X)$ has the FIP, and hence belongs to the class (GC).

PROPOSITION 4.13. *Let X be a finite dimensional space. Let K be an extremally disconnected compact Hausdorff space. Any M-ideal in $C(K, X)$ belongs to the class (GC).*

Proof. We may assume that X has no nontrivial M-ideals (hence no M-summands as X is finite dimensional). This is because if X has nontrivial M-summands, then since X is finite dimensional, there exist subspaces X_1, X_2, ..., X_l of X such that they have no nontrivial M-summands and X is the ℓ^∞ direct sum of these spaces (see [1, Chapter 3]). Now, $C(K, X)$ is the ℓ^∞ direct sum of $C(K, X_i)$'s. And the intersection of an M-deal of the sum to each component space is an M-ideal there. Thus by Corollary 4.8, it is enough to assume that X has no nontrivial M-ideals.

Let $M \subseteq C(K, X)$ be an M-ideal, then $M = \{f \in C(K, X) : f(E) = 0\}$ for a closed set $E \subseteq K$ [1, Corollary 10.2]. As before, let K be homeomorphically embedded in the Stone-Čech compactification $\beta(S)$ of a discrete set S and let $\phi: \beta(S) \to K$ be a continuous retract. Note that via the composition map, E is also closed in $\beta(S)$ (since K is closed) and $C(K, X)$ is embedded into $C(\beta(S), X)$ and this embedding maps $\{f \in C(K, X) : f(E) = 0\}$ onto $\{f \in C(\beta(S), X) : f(E) = 0\}$. So if we now show that $\{f \in C(\beta(S), X) : f(E) = 0\} \in (GC)$, as in the proof of our earlier theorem, the desired conclusion is obtained since ϕ is a retract.

So it is enough to prove that if $M = \{f \in C(\beta(S), X) : f(E) = 0\}$ for a closed set $E \subseteq \beta(S)$, then $M \in (GC)$.

We will use Theorem 1.1(b). Let $\{f_1, f_2, ..., f_n\} \subseteq M$ and let $r_1, r_2, ..., r_n > 0$. As in the proof of [20, Corollary 4.7], for each $s \in S$, choose a pointwise weighted Chebyshev centre $h(s)$ that satisfies

$$\|h(s)\| \leqslant \left(1 + \frac{R}{r_1}\right) \sum_{k=1}^{n} \|f_k(s)\|,$$

where $R = \max\{r_k : k = 1, 2, ..., n\}$. Then h admits a continuous extension to the whole $\beta(S)$; let us call it h again. Since $f_1, f_2, ..., f_n \subseteq M$, $h \in M$ by the above inequality. h is thus a weighted Chebyshev centre. ∎

Using Corollary 2.11, one observes

PROPOSITION 4.14. *Let K be a compact Hausdorff space. Let X be a finite dimensional Banach space such that $C(K, X) \in (GC)$. Then $C(K, X)$ has the n.k.IP for $k = \dim(X) + 1$ and for any $n > k$.*

Proof. By Helly's Theorem [19], X has the $n.k$.IP. By [13], $C(K, X)$ has the $a.n.k$.IP. Since $C(K, X) \in (GC)$, by Corollary 2.11, $C(K, X)$ has the $n.k$.IP. ∎

Remark 4.15. See [20] for examples of spaces X satisfying the above hypothesis. In particular, by [20, Theorem 5.10], for a 2-dimensional X, $C(K, X)$ has the n.3.IP ($n > 3$). Ehrhard Behrends has kindly informed us that, as an application of the above proposition, he has an example of a 3-dimensional space X for which $C(\alpha\mathbb{N}, X)$ fails to be in the class (GC) (here $\alpha\mathbb{N}$ denotes the one point compactification of the set $\mathbb{N}$ of natural numbers). It follows from [20, Theorem 4.7] that $I = \{f \in C(\alpha\mathbb{N}, X) : f(\infty) = 0\} \in (GC)$. It is easy to see that the quotient space $C(\alpha\mathbb{N}, X)/I$ is isometric to X and hence belongs to the class (GC). This shows that membership of the class (GC) is not a 3-space property. We also note that I is a M-ideal in $C(\alpha\mathbb{N}, X)$ and hence a C-subspace (see Proposition 2.8).

We now consider C-subspaces and membership of the class (GC) for spaces of Bochner integrable functions.

As noted in [20], the second author has proved that $L^p(\mu, X) \in (GC)$ when X has the RNP and is 1-complemented in its bidual. Here we reproduce the argument for $1 < p < \infty$.

PROPOSITION 4.16. *Let (Ω, Σ, μ) be a probability space. If X has the RNP and is* 1*-complemented in Y^* then for $1 < p < \infty$, $L^p(\mu, X)$ is* 1*-complemented in $L^q(\mu, Y)^*$ $(1/p + 1/q = 1)$, and hence belongs to the class (GC).*

Proof. We argue as in the proof of [3, Theorem IV.1.1].

Let $P\colon Y^* \to X$ be a projection of norm one.

Define $\hat{P}\colon L^q(\mu, Y)^* \to L^p(\mu, X)$ as follows. For $\Lambda \in L^q(\mu, Y)^*$ define $G\colon \Sigma \to Y^*$ by $G(E)(y) = \Lambda(y\chi_E)$, where χ_E denotes the indicator function of $E \in \Sigma$.

Then G is a countably additive, Y^*-valued measure of bounded variation. And so, $P \circ G$ is a countably additive X-valued measure of bounded variation and by the RNP, we obtain $g\colon \Omega \to X \subseteq Y^*$, the derivative of $P \circ G$ w.r.t. μ.

Now note that the rest of the proof of [3, Theorem IV.1.1] makes no use of the fact that g takes values only in a subspace X of Y^*.

Thus $g \in L^p(\mu, X)$ and $\|g\|_p \leqslant \|\Lambda\|$. Put $\hat{P}(\Lambda) = g$. Clearly $\hat{P}$ is a norm 1 projection. ∎

The following proposition describes another situation where a similar result can be proved without the assumption of the RNP. See [2, Chapter 7, Section 8] for more information on the measurability assumptions considered here.

PROPOSITION 4.17. *Let (Ω, Σ, μ) be a probability space. Suppose X is separable and 1-complemented in X^{**} by a projection P that is weak*-weak universally measurable. Then for $1 \leqslant p < \infty$, $L^p(\mu, X)$ is 1-complemented in $L^q(\mu, X^*)^*$ $(1/p + 1/q = 1)$, and hence, belongs to the class (GC).*

Proof. If $\Lambda \in L^q(\mu, X^*)^*$, then there is a X^{**}-valued w*-measurable g that is a density for Λ (see [3, Section IV.6]). Moreover, the real-valued function $\|g(\cdot)\| \in L^p(\mu)$ (for $p \neq 1$, this is a representation theorem in [8, p. 97] and for the case $p = 1$, one follows the approach of Levin as outlined in [5, p. 200]). Now since P is weak*-weak universally measurable, $P \circ g$ is weakly μ-measurable, and since X is separable, by the Pettis Measurability Theorem [3, Theorem II.1.2], it is strongly μ-measurable. Since g is a density, $P \circ g$ is an L^p function with norm no greater than $\|\Lambda\|$. ∎

Remark 4.18. One familiar example of a space satisfying the above hypothesis, but lacking the RNP, is $L^1([0, 1])$ (see [2, p. 375] for the details).

PROPOSITION 4.19. *Let (Ω, Σ, μ) be a probability space. Let Y be a subspace of X. If for some $1 \leqslant p < \infty$, $L^p(\mu, Y)$ is a C-subspace of $L^p(\mu, X)$, then Y is a C-subspace of X.*

Proof. Let $y_1, y_2, ..., y_n \in Y$ and $x \in X$. Put $g_i = y_i \chi_\Omega$, $i = 1, 2, ..., n$ and $f = x\chi_\Omega$. Then $g_1, g_2, ..., g_n \in L^p(\mu, Y)$ and $f \in L^p(\mu, X)$. Thus there exists $g_o \in L^p(\mu, Y)$ such that $\|g_i - g_o\|_p \leqslant \|g_i - f\|_p$ for all $i = 1, 2, ..., n$. Let $y_o = \int_\Omega g_o \, d\mu$. Then

$$\|y_i - y_o\| = \left\| \int_\Omega g_i \, d\mu - \int_\Omega g_o \, d\mu \right\| \leqslant \|g_i - g_o\|_p \leqslant \|g_i - f\|_p = \|y_i - x\|$$

for all $i = 1, 2, ..., n$. ∎

Remark 4.20. It is clear that all we need in Proposition 4.19 is a set $A \in \Sigma$ such that $0 < \mu(A) < \infty$.

We now prove the converse when Y is a separable subspace. We make a few reductions.

(1) As the properties under consideration involve finitely many elements of $L^p(\mu, X)$ and any $f \in L^p(\mu, X)$, being p-integrable, has σ-finite support, we may assume that μ is σ-finite.

(2) Since the properties we are interested in are invariant under isometries, a normalization trick similar to the one of [13, p. 115] allows us to pass from a σ-finite measure to a probability measure.

(3) Since any two elements of $L^p(\mu, X)$ are identified if they are equal almost everywhere, we may assume μ is complete.

(4) Theorem 4.7 covers the case when μ is purely atomic.

(5) In general, by decomposing μ into purely atomic and non-atomic parts, we see that $L^p(\mu, X)$ is isometric to a ℓ^p-direct sum of discrete and non-atomic parts, thus by Theorem 4.7 again, we may also assume μ itself is non-atomic.

(6) Since only finitely many functions are involved, and they are Bochner p-integrable, they are almost separably valued. Therefore we may assume Σ is countably generated.

By standard techniques in descriptive set theory (see e.g., [18, Section 3.3]), one can actually get a measurable selection as in the proof below in this general set-up. But to avoid the technicalities, we will only prove the result when $\Omega = [0, 1]$, $\Sigma =$ the Lebesgue σ-field and $\mu =$ the Lebesgue measure.

THEOREM 4.21. *Let μ be the Lebesgue measure on $[0, 1]$. If Y is a separable C-subspace of X, then for each $1 \leqslant p < \infty$, $L^p(\mu, Y)$ is a C-subspace of $L^p(\mu, X)$.*

Proof. Let $g_1, g_2, ..., g_n \in L^p(\mu, Y)$ and $f \in L^p(\mu, X)$. By completeness, we may assume that the functions are defined everywhere and being Bochner p-integrable, are Borel measurable.

For $t \in [0, 1]$, consider the multifunction

$$F(t) = \{y \in Y : \|y - g_i(t)\| \leqslant \|f(t) - g_i(t)\|, i = 1, 2, ..., n\}$$

Then $F(t)$ is a nonempty (since Y is a C-subspace of X, considering the points $g_1(t)$, $g_2(t)$, ..., $g_n(t) \in Y$ and $f(t) \in X$) closed convex set in a separable Banach space Y. The graph of F, i.e.,

$$\{(t, y) \colon y \in F(t)\} = \{(t, y) \colon \|y - g_i(t)\| \leqslant \|f(t) - g_i(t)\|, i = 1, 2, ..., n\}$$

is a Borel set (It is in this step that the technical difficulties arise). By the von Neumann selection theorem ([12, Theorem 7.2] or [18, Corollary 5.5.8]), there is a Borel measurable function $g \colon [0, 1] \to Y$ such that

$$\|g(t) - g_i(t)\| \leqslant \|f(t) - g_i(t)\|, \qquad \text{for all} \quad t \in [0, 1], \qquad i = 1, 2, ..., n \tag{3}$$

By the Pettis Measurability Theorem, g is strongly measurable and by (3), $g \in L^p(\mu, Y)$. Also by (3), $\|g - g_i\|_p \leqslant \|f - g_i\|_p$ for all $i = 1, 2, ..., n$. ∎

COROLLARY 4.22. *Let μ be the Lebesgue measure on $[0, 1]$. Let X^* have the RNP and let Y be a separable C-subspace of X^*. Then for each $1 \leqslant p < \infty$, $L^p(\mu, Y) \in (GC)$.*

Proof. By the above theorem, $L^p(\mu, Y)$ is a C-subspace of $L^p(\mu, X^*)$. Under the hypothesis, for $1 < p < \infty$, $L^p(\mu, X^*)$ is a dual space, while $L^1(\mu, X^*)$ is 1-complemented in its bidual [14] and hence belongs to the class (GC). ∎

QUESTION 4.23. *If $Y \in (GC)$ and has the RNP, does there exist a dual space X^* with the RNP such that Y is a C-subspace of X^*?*

COROLLARY 4.24. *Let μ be the Lebesgue measure on $[0, 1]$. Suppose X^* has the RNP. Let $Z \subseteq Y \subseteq X^*$ be such that Z is w*-closed in X^* and Y is a separable C-subspace of X^*. Then for each $1 \leqslant p < \infty$, $L^p(\mu, Y/Z) \in (GC)$.*

Proof. Since Z is w*-closed in X^*, it is proximinal and X^*/Z is a dual space with the RNP. By Proposition 4.4, Y/Z is a separable C-subspace of X^*/Z. Now use the above corollary. ∎

Remark 4.25. If $Z \subseteq Y \subseteq Y^{**}$, then Z is w*-closed in Y^{**} if and only if Z is reflexive. Thus the above corollary is in a larger framework.

Remark 4.26. The second author has recently succeeded in showing that if $L^1(\mu, X^*) \in (GC)$ and $M \subseteq X^*$ is a weak* closed subspace having the RNP then $L^1(\mu, X^*/M) \in (GC)$. In particular if $L^1(\mu, X) \in (GC)$ and $M \subseteq X$ is a reflexive subspace then $L^1(\mu, X/M) \in (GC)$ (See [17]).

Remark 4.27. If Y is a semi-L-summand in X, then it is easy to see that $L^1(\mu, Y)$ is a semi-L-summand in $L^1(\mu, X)$ (see [16]) and hence is a C-subspace without any further assumptions.

We conclude with another instance where Chebyshev centres are preserved by Bochner spaces.

If X has the 3.2.IP, then it has Chebyshev centre for 3 elements (see [20] for a space with a 3 element set which has no Chebyshev centre). It is proved in [11] that for any (Ω, Σ, μ), $L^1(\mu, X)$ has the 3.2.IP, and hence Chebyshev centre exists for all sets of 3 elements.

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